Maximum Efficiency of Energy Release in Spherical Collapse*

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The question of the maximum amount of energy which can be radiated by a collapsing spherical star is reexamined. On Newtonian theory, gravitational energy is negative and unbounded below, so that unlimited amounts of energy can be released. It is shown that, in the relativistic collapse of a star with non-negative energy density, self-closure always takes place before the star can release 100% of its initially positive mass energy. Moreover, under physically reasonable restrictions on the pressure, the 100% upper limit can be approached only if the star happens to pass through a very special and improbable momentarily static configuration first considered by Zel'dovich. It is concluded that for normal spherical collapse the efficiency of energy release must be low.

I. INTRODUCTION

Early attempts to explain the prodigious energy output of quasistellar sources in terms of the collapse of massive superstars[1] foundered on two major difficulties; the relatively brief time span of the active phase of collapse, and the lack of an efficient mechanism to convert kinetic energy of fall into outgoing energy. Rough estimates[1] indicated that the efficiency (energy release/original mass energy) for spherical collapse is unlikely to exceed a few percent; however, detailed integrations of the relativistic hydrodynamical equations with allowance for energy emission have yet to be carried out.

Recently, Dyson[2] raised the question whether an upper bound on the efficiency can be inferred from purely energetic considerations. His conclusion was that a collapsing spherical star is capable, in principle, of releasing up to 100% of its mass energy, though the theoretical upper limit is only approachable under special and improbable circumstances.

This conclusion will be reaffirmed in this paper, but we shall show that Dyson's argument needs modification and that the condition for approaching 100% efficiency is actually much more restrictive than previously supposed.

The condition is simply that each layer of the star should be momentarily and simultaneously brought to rest just outside the Schwarzschild radius $r=2m(r,t)$ corresponding to the mass interior to it. That this leads to efficiencies arbitrarily close to 100% is easily understood from quite simple considerations. If a collection of baryons, whose mass is ΔM when dispersed at infinity, is lowered quasistatically in the field of a spherical mass $m(\Delta M \ll m)$ and assembled to form a thin spherical shell of radius r, its contribution to the total gravitational mass is only $\Delta m=\Delta M(1-2m/r)^{1/2}$, the reduction arising from the loss of potential energy. It follows that the gravitational mass of an arbitrarily large number of baryons can be brought arbitrary close to zero if they are assembled as a series of momentarily static shells with radii $r=2m(r,t)+\epsilon(r,t)$ $(\epsilon\to 0+)$.

In the following sections we shall prove that this very special and artificial distribution (first discussed by Zel'dovich[3]) is essentially the *only* one which yields gravitational masses arbitrary close to zero for material with positive energy-density and pressure. It is therefore safe to conclude that the efficiency in any normal situation of spherical collapse must fall far short of the theoretical 100% limit.

II. EFFICIENCY CANNOT EXCEED 100%

We begin by showing that a static or nonstatic spherical distribution whose local energy-density is everywhere non-negative cannot develop a negative gravitational mass. While this result is often treated as "obvious," an explicit proof has not been previously given to our knowledge.[4]

In terms of curvature coordinates,[5] the spherical line element may be written

$$ds^2=[1-2m(r,t)/r]^{-1}dr^2+r^2d\Omega^2 -[1-2m(r,t)/r]e^{2\psi(r,t)}dt^2. \quad (1)$$

These coordinates remain nonsingular so long as r remains spacelike and a monotonic function of radial arc length, and both of these conditions will obtain so long as

$$2m(r,t)<r. \quad (2)$$

From the field equation $G_4{}^4=-8\pi T_4{}^4$, we find[5]

$$\partial m(r,t)/\partial r=-4\pi r^2T_4{}^4\geq 0\,, \quad (3)$$

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[1] *Quasi-Stellar Sources and Gravitational Collapse*, edited by I. Robinson, A. Schild, and E. L. Schucking (University of Chicago Press, Chicago, 1964).

[2] F. J. Dyson, Comments Astrophys. Space Phys. 1, 75 (1969).

[3] Ya. B. Zel'dovich, Zh. Eksperim. i Teor. Fiz. 42, 641 (1962) [Soviet Phys. JETP 15, 446 (1962)].

[4] See, however, D. R. Brill and S. Deser, Ann. Phys. (N. Y.) 50, 548 (1968).

[5] See, e.g., J. L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960), Chap. 7.

since the positive-energy condition ($T_\nu{}^\mu u_\mu u^\nu \geq 0$ for all timelike vectors u^μ) implies $T_4{}^4 \leq 0$. We write $r=R(t)$ for the boundary with the exterior vacuum, so that $m[R(t),\, t] \equiv m$ is the externally observed gravitational mass.

Now suppose m is negative. Then (2) is trivially satisfied for $r \geq R(t)$, showing that r is monotonically increasing in the exterior region, and decreases (at least initially) as one moves inwards from the surface. The inequality (3) ensures that (2) remains valid with continuously increasing depth from the surface, so that r continues to decrease and t remains meaningful. We can therefore integrate (3) inwards to the center, where we necessarily encounter a negative-mass singularity: $m(0,t) \leq m < 0$. This contradiction establishes that, actually, $m \geq 0$.

It should be noted that our result $m \geq 0$ presupposes *finite* exterior time t, and is therefore a restriction only on the *externally* measurable gravitational mass. In fact, general relativity imposes no restriction on the amount of radiation which can leave a star's surface *after* it has collapsed inside the Schwarzschild radius[6]; however, all this energy is trapped and cannot reduce the external mass. We could sum this up by saying that self-closure will always act so as to prevent the development of negative mass in relativistic collapse.

III. ONLY THE ZEL'DOVICH DISTRIBUTION GIVES 100% EFFICIENCY

The number of baryons in the star is

$$A = \int_{\Sigma_3} n\,|u^\mu N_\mu|\,d\Sigma\,,$$

where $n(r,t)$ is the baryon density, u^μ the four-velocity, and the integral is taken over any complete spacelike section Σ_3 with unit timelike normal N_μ. If Σ_3 is identified with $t=$const, we have $|N_4|\,d\Sigma = e^\psi 4\pi r^2 dr$. Hence $A = A[R(t),\, t]$, where we have defined

$$A(r,t) \equiv \int_0^r n u^4 e^\psi 4\pi r^2 dr\,. \tag{4}$$

We now make four assumptions, of which only the first two are essential; assumptions (iii) and (iv) are introduced for simplicity and can be avoided by a more elaborate argument.

(i) The star evolves from a nonsingular initial state. (This means that initially the star's surface is represented by a timelike curve in quadrant I of the Kruskal diagram shown in Fig. 1.)

(ii) The local energy-density is non-negative. The fluid pressure is non-negative and does not exceed the energy density.

[6] For a description of the collapse of a radiating star within its Schwarzschild radius, see W. Israel, Phys. Letters **24A**, 184 (1967).

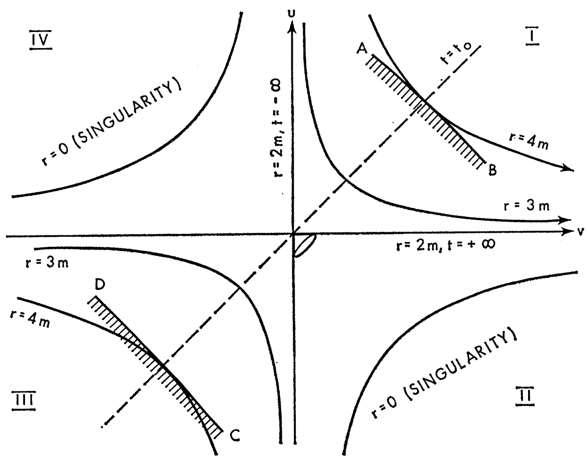

FIG. 1. Kruskal diagram for Schwarzschild's vacuum space-time using lightlike axes u,v. Quadrant I represents "our" universe, quadrant III a different asymptotically flat universe joined to ours by an Einstein-Rosen bridge. The timelike curves AB and CD represent two possible histories for the boundary of one of the stellar models discussed in Sec. IV. Shading on the curves distinguishes the star's interior.

(iii) The hypersurface $t=$const is globally spacelike and can be extended to the center.[7] This implies that (2) is satisfied for all r and that r is monotonic.

(iv) Any radiation produced has already escaped from the star, so that we are dealing purely with a simple fluid having energy density $\mu(r,t)$ and pressure $P(r,t)$.

From (3) and (4), we obtain for the gravitational mass

$$m = \int_0^{R(t)} (-\mu u^4 u_4 + P u^1 u_1) 4\pi r^2 dr$$

$$= \int_0^A \left(\frac{\mu}{n}\right) e^{-\psi} |u_4|\, dA(r,t) + 4\pi \int_0^{R(t)} P u^1 u_1 r^2 dr\,.$$

The second integral is non-negative, and vanishes for a momentarily static configuration. In the positive definite integrand of the first integral we can set

$$e^{-\psi}|u_4| = [1 - 2m(r,t)/r + (u^1)^2]^{1/2}, \tag{5}$$

[7] This assumption may not be valid during the final stages of collapse to the Schwarzschild radius. Hence it is preferable to work with null hypersurfaces of constant retarded time u (u is constant along outgoing radial light rays), for which this difficulty does not arise. The appropriate formalism (allowing at the same time for outflowing radiation) has been set up by W. C. Hernandez and C. W. Misner, Astrophys. J. **143**, 452 (1966). From Eq. (5) of this reference and the discussion following Eq. (41) it follows that Schwarzschild's radial coordinate r decreases monotonically as one moves outwards relatively to the matter at constant u, provided $\mu + P > 0$. Then, from Eqs. (B8) and (B13) and the discussion following Eq. (14), one can easily set up an argument paralleling the above, but assuming only our conditions (i) and (ii), and arrive at precisely the same conclusion. (Note that when each part of the star is just outside its Schwarzschild radius there is effectively no difference between "simultaneity" as defined in terms of t or of u.)

an immediate consequence of $g^{\alpha\beta}u_\alpha u_\beta=-1$. By choosing the expression (5) (more accurately, its mass average over the star) sufficiently small, and *only* by so doing, we can reduce the first integral to arbitrarily small values for any given number of baryons A and any given equation of state.

The necessary and sufficient condition that $m\approx 0$ for fixed A is therefore essentially that the star pass through a momentarily static configuration with mass distribution given by the Zel'dovich condition[3] $r=2m(r,t)+\epsilon(r,t)$ $(\epsilon\rightarrow 0+)$, i.e., $\mu\approx 1/8\pi r^2$. This distribution could be truncated internally so as to form a hollow shell[8]; if it is continued inwards to the center, $\epsilon(r,t)$ must be adjusted carefully to avoid a singularity there.[3]

IV. NATURE OF "UNIFORM CONFIGURATIONS"

An apparent counterexample to the result just proven is to be found in Ref. 9, where a sequence of momentarily static configurations is exhibited whose mass tends to zero for arbitrarily large A, and which have *uniform* density. However, as we shall now show, these models have unusual properties which make them inapplicable in the present context.

The momentarily static configurations constructed in Ref. 9 are characterized by $\mu(r,t_0)=\mu_0=\text{const}$, so that $m(r,t_0)=\frac{4}{3}\pi\mu_0 r^3$. Introducing a new radial coordinate χ, defined by

$$r=a\sin\chi$$

into (1), where $a\equiv[(8/3)\pi\mu_0]^{-1/2}$, we find

$$(ds^2)_{t=t_0}=a^2(d\chi^2+\sin^2\chi\, d\Omega^2)-e^{2\psi}\cos^2\chi\, dt^2, \qquad (6)$$

showing that $a\chi$ measures radial proper distance from the center. The boundary of the configuration is given by

$$\chi=\chi_0,\quad r=R(t_0)=a\sin\chi_0,\quad m=\tfrac{1}{2}a\sin^3\chi_0. \qquad (7)$$

The authors of Ref. 9 now argue that if χ_0 is allowed to increase towards π, the baryon number

$$A=4\pi a^3\int_0^{\chi_0} n\sin^2\chi\, d\chi$$

increases towards a finite limit, whereas (7) shows that both the radius and the gravitational mass sink to zero.

To understand the origin of this result, we must examine more closely the nature of the configurations with $\chi_0>\frac{1}{2}\pi$. These models can be interpreted in two possible ways.

[8] For *thin* shells in equilibrium the fact that $m\rightarrow 0$ in the ultrarelativistic limit has been noted and discussed in some detail by J. E. Chase [Nuovo Cimento (to be published)].

[9] B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, *Gravitation Theory and Gravitational Collapse* (University of Chicago Press, Chicago, 1965), Chap. 8.

According to the first interpretation, the star is (at least initially) an accessible component of "our" universe with boundary represented by a curve, such as AB, in quadrant I of Fig. 1. However, if such a configuration (with $\chi_0>\frac{1}{2}\pi$) were truly a distribution of uniform density μ_0, as we might at first suppose, then we run into the following difficulty: Adding a spherical shell of uniform density μ_0 should lead to a new "uniform" configuration whose radius R is both *larger* than the original (since it extends further into the exterior Schwarzschild space where r is monotonically increasing) and also *smaller* (since it corresponds to a larger A and larger χ_0).

At the root of this difficulty is clearly the fact that r changes abruptly from decreasing outwards to increasing outwards at the boundary of a configuration with $\chi_0>\frac{1}{2}\pi$, and this indicates the presence of a shell of mass.

We adopt θ, ϕ, and proper time τ as intrinsic coordinates θ^a of the boundary, so that the intrinsic three-metric is

$$g_{ab}d\theta^a d\theta^b=R^2d\Omega^2-d\tau^2.$$

The surface energy three-tensor S_{ab} of the shell is given in terms of the jump $\gamma_{ab}=K_{ab}{}^+-K_{ab}{}^-$ of the extrinsic curvature by[10]

$$-8\pi S_{ab}=\gamma_{ab}-g_{ab}g^{cd}\gamma_{cd}.$$

We thus derive the surface density

$$S_{\tau\tau}=-(1/4\pi R^2)\gamma_{\theta\theta}.$$

A straightforward calculation yields for the extrinsic curvatures of the imbeddings in the interior space (6) and the exterior Schwarzschild space at the moment of the time symmetry $t=t_0$

$$K_{\theta\theta}{}^-=a\sin\chi_0\cos\chi_0,$$

$$K_{\theta\theta}{}^+=R(1-2m/R)^{1/2}=a\sin\chi_0|\cos\chi_0|.$$

Substitution into (8) shows that $S_{\tau\tau}<0$ if $\chi_0>\frac{1}{2}\pi$.

We conclude that the "uniform" configurations with $\chi_0>\frac{1}{2}\pi$, if regarded as part of "our" universe, are actually encased in a layer of negative mass.

The second interpretation of the $\chi_0>\frac{1}{2}\pi$ models avoids this particular difficulty. The boundary is considered to be a curve, such as CD, in quadrant III of Fig. 1. This makes the gradient of r continuous across the boundary. However, the exterior vacuum region now includes two singular curves $r=0$ and both past and future event horizons $r=2m$, $t=\pm\infty$. To an external observer such an object would appear simply as a "black hole" which has existed in that form for all time, and it is specifically excluded from the considerations of Sec. III by our assumption of nonsingular initial conditions.

[10] W. Israel, Nuovo Cimento **44B**, 1 (1966); **48B**, 463 (1967).